Implementation in the Cake-cutting problem [*†]

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Abstract

This paper formalizes the relationship between fractional cake allocations — partitions of unity — and lotteries over deterministic partitions. The former is an insightful embedding used in mathematical fair division, while the latter is a common tool for convexifying a set of deterministic allocations. By proving that these two objects are linear utility-equivalent, I provide an implementation foundation for a mathematical representation commonly used in fair division. In addition, I show that every simple partition of unity belongs to the convex hull of degenerate partitions of unity, which means it is implementable by a lottery with finite support independently of the preference domain.

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1. Introduction

In many allocation problems, it is a non-trivial quest to find a good way of modeling the space of feasible allocations. The trade-offs are particularly demanding when the space of allocations is large. On the one hand, modeling tools benefit from being realistic, that is reflecting the real-world procedures. Another attractive attribute is being intuitive and easy to interpret when communicating ideas to the members of the science.

On the other hand, models are limited by the available technical toolkits and the established ways of thinking about problems. To gain a chance of obtaining results of interest, a researcher has to carefully think about the assumption-conclusion trade-offs of different mathematical structures. Oftentimes, striking the right balance between generality, interpretability and technical stringency makes the search for a useful set-up challenging.

Cake-cutting literature is well aware of the struggle. In pursuit of fairness, efficiency, and sometimes both, scholars came up with a great way to model the space of feasible allocations in the problem by taking it as a space of partitions of unity. Embedded in the right topological vector space, the technical potency of the tool is undisputable.

However, the interpretability and implementability of the partitions of unity is somewhat less satisfactory. In this project, I provide arguments in support of partitions of unity against that critique.

The main objective of the cake-cutting problem is to distribute a comestible among finitely many agents. The primitive set of allocations of this problem is nothing but the set of partitions of the comestible: consumption bundles, each containing a slice of cake, that jointly respect the total supply constraint.

Mathematically, such a set is unruly and technically demanding to work with. On top of that, there is a subtle flavor of indivisibility, even if the comestible itself is perfectly divisible (for instance, when agents' preferences are identical Dirac measures). As is well known in the literature of fair division with indivisible goods, strong notions of fairness are impossible to satisfy if insisting on allocating deterministically.

Luckily, the impossibility results can be avoided by allowing for randomization over the set of deterministic feasible allocations. Equipped with the viewpoint of randomization, I provide a series of results that help partitions of unity stand on firmer ground with regard to both interpretability and implementability.

The paper is structured as follows. Section 2 presents a brief overview of the literature. Section 3 introduces the language.

In Section 4, I show that the set of partitions of unity is utility-equivalent to the set of probability measures on the space of deterministic partitions. The two directions of the main result require different approaches.

To construct a utility-equivalent lottery for every partition of unity, I use a common randomization device. For a uniform distribution on $[0, 1]$, I construct a jointly measurable map that specifies slices of cake that agents get for every realization of the state. The push-forward measure of such a map has the desirable property.

The construction of a utility-equivalent partition of unity for any lottery over deterministic

partitions is surprisingly more challenging and is done only for the class of standard Borel spaces. The original approach that highlights the main idea is presented in the sketch of proof of the result in Section 4.2. When tested with ChatGPT-6, the proof contained a measurability mistake. The corrected version of the proof is in Appendix A and was generated by ChatGPT-6.

In Section 5, I move from a weaker notion of utility equivalence to a stronger notion of direct implementability. In the spirit of Birkhoff–von Neumann theorem, I show that every partition of unity of simple functions belongs to the convex hull of the partitions of unity of indicator functions. The main implication of the result is that independently of the preference domain, every such partition of unity can be implemented by a lottery over deterministic partitions with finite support.

2. Literature Review

The famous literature on cake cutting traces back to Steinhaus (1948), who formalized the problem known as the cake-cutting problem, starting the literature on fair division. But this particular definition of fairness, that is, envy-freeness and Pareto efficiency, has not found its way into the literature until Weller (1985). He proved the existence of an EF and PE allocation in the problem of allocating a measurable space $(\Omega, \mathcal{B})$, where agents' preferences are atomless measures on this space. He also showed that such an allocation is supported as a price equilibrium in the exchange economy with equal incomes.

Akin (1995) builds on Weller and extends the result to the settings where measures are not necessarily atomless. His notion of envy-freeness is also more general, allowing for assigning non-equal weights to agents' evaluations.

Weller's allocation is a partition, since he assumes atomless measures. Akin's allocation is a partition of unity, since he does not. Both of them give a way to construct such an allocation, though: it is supported as a price equilibrium for certain prices/budgets. My contribution to the literature is that even when measures are not atomless, I show that the allocation in question is a lottery over partitions.

Berliant et al. (1992) prove the existence of an allocation satisfying PE and the stronger notion of group envy-freeness, but the comestible in the paper is a compact subset of $\mathbb{R}^k$ and agents' utilities are absolutely continuous with respect to Lebesgue measure. They also prove that under certain assumptions, envy-freeness implies group envy-freeness and efficiency. For example, their sufficient conditions are applicable in the problem of allocating sub-intervals of $[0, 1]$ as in Woodall (1980), and the existence of an EF allocation has already been proved.

Maccheroni and Marinacci Maccheroni and Marinacci (2003) extend classical cake-cutting results to environments in which agents have decreasing marginal evaluations. They establish existence for fair division in this more general setting.

Hüsseinov and Sagara Hüsseinov and Sagara (2013) model a commodity as a measurable space. They relax the convexity assumption of the preferences and prove the existence of a PE and EF allocation under certain assumptions on the utility function of each individual: uniform continuity, strict monotonicity with respect to set inclusion order, and closed partition

matrix range of the utility functions.

3. Setup

Let $(L, \mathcal{B})$ be a measurable space, where L is a set of divisible resources, which I refer to as the *cake*, and $\mathcal{B}$ is a σ-algebra of measurable subsets of L. Let $N = \{1, \ldots, n\}$ be a finite set of agents. Each agent $i \in N$ has preferences over measurable subsets of L represented by a finite positive measure μ_i on $(L, \mathcal{B})$.

3.1 Deterministic allocations

An individual deterministic consumption bundle is a measurable subset of L. Thus, the space of individual consumption bundles is

$$Y = \mathcal{B}.$$

A deterministic allocation is a measurable partition $F = (F_1, \ldots, F_n)$ of L, where F_j is the subset allocated to agent j. The set of feasible deterministic allocations is

$$X = \left\{ (F_1, \ldots, F_n) \in Y^n : \bigcup_{j \in N} F_j = L, \quad F_i \cap F_j = \varnothing \text{ for } i \neq j \right\}.$$

I equip Y with the σ-algebra generated by evaluations of measurable sets under finite positive measures. Let $\mathcal{M}_f(L)$ denote the collection of all finite positive measures on $(L, \mathcal{B})$. For each $\mu \in \mathcal{M}_f(L)$, define

$$q_\mu : Y \longrightarrow \mathbb{R}, \qquad q_\mu(F) = \mu(F).$$

The σ-algebra on Y is

$$\mathcal{M}_Y = \sigma\left(q_\mu : \mu \in \mathcal{M}_f(L)\right).$$

I equip X with the trace of the product σ-algebra:

$$\mathcal{M}_X = \mathcal{M}_Y^{\otimes n}\Big|_X.$$

Equivalently,

$$\mathcal{M}_X = \sigma\left(F \longmapsto \mu(F_j) : \mu \in \mathcal{M}_f(L),\ j \in N\right).$$

The inclusion of all finite positive measures in the definition of $\mathcal{M}_Y$ is important. In particular, for every $x \in L$, the Dirac measure δ_x is a finite positive measure, and hence the point-membership map

$$F \longmapsto \mathbf{1}_{\{x \in F_j\}}$$

is $\mathcal{M}_X$-measurable for every $j \in N$.

Lotteries over deterministic allocations A lottery over deterministic allocations is a probability measure on the measurable space $(X, \mathcal{M}_X)$. I denote the set of all such lotteries by $\Delta(X)$.

3.2 Preferences

Fix a family of preference measures $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. Agents are risk neutral, and their preferences over lotteries admit an expected utility representation.

For every lottery $p \in \Delta(X)$, the utility of agent $i \in N$ is defined by

$$U_i(p) = \int_X \mu_i(F_i)\, dp(F).$$

More generally, the expected utility of agent i from the consumption bundle assigned to agent j is

$$U_i(p_j) = \int_X \mu_i(F_j)\, dp(F).$$

These integrals are well-defined because the mappings $F \mapsto \mu_i(F_j)$ are measurable and bounded by $\mu_i(L)$.

I denote the matrix of expected utilities by

$$V_{\boldsymbol{\mu}}(p) = \left(\int_X \mu_i(F_j)\, dp(F) \right)_{i,j \in N}.$$

3.3 Random partitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 3.1. A **random partition** of L is a family $A = (A_1, \ldots, A_n)$ of sets in $\mathcal{B} \otimes \mathcal{F}$ satisfying

$$\bigcup_{j \in N} A_j = L \times \Omega, \qquad A_i \cap A_j = \varnothing \quad \text{for } i \neq j.$$

For each $\omega \in \Omega$, define the section

$$F_j^\omega = \{x \in L : (x, \omega) \in A_j\}.$$

Each F_j^ω belongs to $\mathcal{B}$, and the family

$$F^\omega = (F_1^\omega, \ldots, F_n^\omega)$$

is a deterministic partition of L. Thus, a random partition determines a mapping

$$\Phi_A : \Omega \longrightarrow X, \qquad \Phi_A(\omega) = F^\omega.$$

The map Φ_A is measurable (proof in the Appendix A) and induces a lottery on X through the pushforward measure

$$p_A = (\Phi_A)_\# P,$$

defined by

$$p_A(E) = P\Big(\Phi_A^{-1}(E)\Big), \qquad E \in \mathcal{M}_X.$$

I denote by $\Delta_{\mathrm{rp}}(X)$ the collection of lotteries induced by random partitions.

3.4 Partitions of unity

Definition 3.2. A **partition of unity** on L is a family $f = (f_1, \ldots, f_n)$ of $\mathcal{B}$-measurable functions $f_j : L \to [0,1]$ satisfying

$$\sum_{j \in N} f_j(x) = 1 \qquad \text{for every } x \in L.$$

Denote the set of all partitions of unity by

$$Z = \left\{ (f_1, \ldots, f_n) : f_j : L \to [0,1] \text{ is measurable}, \sum_{j \in N} f_j = 1 \right\}.$$

The utility of agent i from a partition of unity $f \in Z$ is defined by

$$U_i(f) = \int_L f_i, d\mu_i.$$

More generally, the utility of agent i from the fractional consumption bundle assigned to agent j is

$$U_i(f_j) = \int_L f_j, d\mu_i.$$

Correspondingly, the utility matrix of a partition of unity $f \in Z$ is

$$V_{\boldsymbol{\mu}}(f) = \left(\int_L f_j \, d\mu_i \right)_{i,j \in N}.$$

A partition of unity is called **simple** if each f_j is a simple function. Denote the set of simple partitions of unity by Z^0.

A partition of unity is called **deterministic** if each f_j takes values in $\{0,1\}$. Such partitions of unity are in one-to-one correspondence with deterministic allocations in X.

3.5 Utility equivalence

Definition 3.3. A lottery $p \in \Delta(X)$ and a partition of unity $f \in Z$ are **utility-equivalent** with respect to $\boldsymbol{\mu}$ if

$$\int_X \mu_i(F_j) \, dp(F) = \int_L f_j \, d\mu_i \qquad \text{for every } i, j \in N.$$

Equivalently, $V_{\boldsymbol{\mu}}(p) = V_{\boldsymbol{\mu}}(f)$.

Definition 3.4. A lottery p and a partition of unity f are **universally utility-equivalent** if the same equalities hold for every finite positive measure $\mu \in \mathcal{M}_f(L)$ and every $j \in N$:

$$\int_X \mu(F_j)\, dp(F) = \int_L f_j\, d\mu \qquad \text{for every } \mu \in \mathcal{M}_f(L),\ j \in N.$$

Universal utility equivalence is therefore independent of the particular family of agents' preference measures.

4. Equivalence Results

Randomization devices are widely used in economics, notably in the study of correlated equilibria in game theory and social choice. To construct lotteries over deterministic allocations of the cake, I turn to a related device: **random partitions** (RP), which assign a deterministic partition of the cake to each realization of an underlying random state.

It turns out there is a tight connection between random partition and partitions of unity. This connection relies on an insightful intermediate step, which is to show that every measurable random partition A induces a lottery $p_A \in \Delta(X)$. After that, my first two results jointly establish that lotteries induced by measurable random partitions are universally utility-equivalent to partitions of unity.

The abovementioned connection can be summarized as follows:

$$\text{Measurable random partition} \xrightarrow{\text{Pushforward}} \text{RP-induced lottery} \overset{\text{Universal utility equivalence}}{\longleftrightarrow} \text{Partition of unity}$$

Notice that not every lottery over deterministic allocations is induced by a measurable random partition. This observation does not concern the issue of implementation. Modeling the space of feasible allocations as a set of partitions of unity is implementation-justified, since every partition of unity is utility-equivalent to a lottery over deterministic partitions, and on top of that, the lottery is generated by a random partition.

Alas, when reasoning about Pareto efficiency, one might question whether working with partitions of unity turns the notion into a constrained efficiency notion. I address this concern in the section 4.2 and claim that for standard Borel spaces, for every fixed preference profile μ it is not the case.

4.1 Universal utility equivalence of random partitions and partitions of unity

The first result establishes that for every RP-induced lottery, there exists a universally utility-equivalent partition of unity.

Theorem 4.1. *Every induced lottery $p_A = (\Phi_A)_{\#}P$ is universally utility-equivalent to some partition of unity $f \in Z$.*

Proof. Define

$$f_j(x) = \int_\Omega \mathbf{1}_{A_j}(x, \omega)\, dP(\omega), \qquad j \in N.$$

By Tonelli's theorem, each f_j is $\mathcal{B}$-measurable and takes values in $[0,1]$. Since A partitions $L \times \Omega$, for every $x \in L$,

$$\sum_{j \in N} f_j(x) = \int_\Omega \sum_{j \in N} \mathbf{1}_{A_j}(x,\omega)\, dP(\omega) = 1.$$

Thus, $f = (f_1, \ldots, f_n) \in Z$.

Finally, let $\mu \in \mathcal{M}_f(L)$ and $j \in N$. By the definition of the pushforward measure and Tonelli's theorem,

$$\begin{aligned}
\int_X \mu(F_j)\, dp_A(F) &= \int_\Omega \mu(F_j^\omega)\, dP(\omega) \\
&= \int_\Omega \int_L \mathbf{1}_{A_j}(x,\omega)\, d\mu(x)\, dP(\omega) \\
&= \int_L \int_\Omega \mathbf{1}_{A_j}(x,\omega)\, dP(\omega)\, d\mu(x) \\
&= \int_L f_j(x)\, d\mu(x).
\end{aligned}$$

I conclude by observing that this equality holds for every finite positive measure μ and every $j \in N$. □

Note that the interpretation of a partition of unity is as follows: for every point on a cake $x \in L$, the value $f_j(x)$ represents the probability that agent j is allocated x.

Next comes the converse statement.

Theorem 4.2. *For every partition of unity $f \in Z$, there exist a probability space $(\Omega, \mathcal{F}, P)$ and a random partition A such that the induced lottery p_A is universally utility-equivalent to f.*

Proof. Let $f = (f_1, \ldots, f_n) \in Z$. Take $\Omega = [0,1), \mathcal{F} = \mathcal{B}([0,1)), P = \lambda$, where λ denotes Lebesgue probability measure.

For each $j \in N$, define the cumulative functions

$$S_0(x) = 0, \qquad S_j(x) = \sum_{k=1}^{j} f_k(x).$$

Since $f \in Z$, each S_j is measurable and

$$0 = S_0(x) \leq S_1(x) \leq \cdots \leq S_n(x) = 1$$

for every $x \in L$.

Define

$$A_j = \{(x,\omega) \in L \times [0,1) : S_{j-1}(x) \leq \omega < S_j(x)\}, \qquad j \in N.$$

Observe that, for each $x \in L$, the set A_j records precisely those states ω in which the point x is allocated to one of the agents $\{1, \ldots, j\}$, but not to any of the agents $\{1, \ldots, j-1\}$. Indeed, the first condition corresponds to $\omega < S_j(x)$, while the second corresponds to $\omega \geq S_{j-1}(x)$. Both of these sets are measurable since S_j is measurable.

Therefore, each A_j is a finite intersection of measurable sets. Moreover, for every $x \in L$, the half-open intervals

$$[S_{j-1}(x), S_j(x)), \qquad j \in N,$$

are pairwise disjoint and partition $[0, 1)$. Hence, $A = (A_1, \ldots, A_n)$ is a random partition of L.

Recall from Section 3.3 that this random partition induces a lottery $p_A \in \Delta(X)$.

For every $x \in L$ and $j \in N$,

$$\begin{aligned} P\Big(\{\omega : (x, \omega) \in A_j\}\Big) &= \lambda\Big([S_{j-1}(x), S_j(x))\Big) \\ &= S_j(x) - S_{j-1}(x) \\ &= f_j(x). \end{aligned}$$

Consequently, for every $\mu \in \mathcal{M}_f(L)$ and $j \in N$,

$$\begin{aligned} \int_X \mu(F_j)\, dp_A(F) &= \int_L P\Big(\{\omega : (x, \omega) \in A_j\}\Big)\, d\mu(x) \\ &= \int_L f_j(x)\, d\mu(x). \end{aligned}$$

□

4.2 Utility equivalence of lotteries and partitions of unity

To deliver the next result, I take $(L, \mathcal{B})$ to be a standard Borel space. I show that for any fixed family of preference measures $\boldsymbol{\mu}$, every lottery in $\Delta(X)$ is utility-equivalent to some partition of unity. It turns out that this equivalence cannot, in general, be strengthened to universal utility equivalence: there exist lotteries for which no universally utility-equivalent partition of unity exists. The counter-example in Appendix B is generated by ChatGPT-6 and verified by me.

Theorem 4.3. *Suppose that $(L, \mathcal{B})$ is a standard Borel space. Fix a family of finite positive preference measures $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. Then, for every lottery $p \in \Delta(X)$, there exists a partition of unity $f \in Z$ that is utility-equivalent to p.*

Sketch. The main idea of the proof is close to the central idea of the proof of the theorem 4.1. For a point $x \in L$ and a deterministic partition F, define

$$g_j(x, F) = \mathbf{1}_{F_j}(x)$$

Observe that these functions are measurable in each argument, but the joint measurability is non-trivial. For this sketch, I assume that g_j is jointly measurable.

Define the partition of unity by

$$f_j(x) = \int_X g_j(x, F)\, dp(F)$$

The role of g_j is to keep track of the points of the cake that are allocated to agent j under

partition F. Observe that in the theorem 4.1, the map $\mathbf{1}_{A_j}(x,\omega)$ is playing a similar role, but the partition changes with ω.

The key idea that is identical in both proofs is to interpret $f_j(x)$ as the probability that a point x is allocated to agent j under a lottey p on the space of deterministic partitions.

Applying Tonelli's theorem proves the theorem

$$\begin{aligned}\int_L f_j(x)\,d\mu_i(x) &= \int_L \int_X g_j(x,F)\,dp(F)\,d\mu_i(x)\\ &= \int_X \int_L g_j(x,F)\,d\mu_i(x)\,dp(F)\\ &= \int_X \int_L \mathbf{1}_{F_j}(x)\,d\mu_i(x)\,dp(F)\\ &= \int_X \mu_i(F_j)\,dp(F).\end{aligned}$$

Joint measurability of the map g is a delicate point of the approach. The full proof, delegated to the Appendix A, approximates the family $\{g_j\}_{j\in n}$ with a sequence of jointly measurable maps. The sequencce is convergent by a martingale convergence theorem. The limit h_j is equal to g_j almost surely with respect to a measure m, where $m = \sum_{j=1}^n \mu_j$. Since for every j μ_j is absolutely continuous wrt to m, the limit h_j is equal to g_j almost surely with respect to μ_j.

□

5. Convex hull of deterministic partitions of unity

The previous section provides a utility-equivalence justification of employing partitions of unity as a useful modeling tool. In this section, I address a slightly different issue that arises in the cake cutting literature.

Before I can highlight the issue, I need to introduce some language and invoke several results from Dvoretzky, Wald, and Wolfowitz **?**, later referred to as DWW.

Given a partition of unity $f \in Z$, recall that $V_{\boldsymbol{\mu}}(f)$ denotes the $n \times n$ matrix of agents' utilities from their own allocations and the allocations of every other agent:

$$V_{\boldsymbol{\mu}}(f) = \left(\int_L f_j\,d\mu_i\right)_{i,j\in N}.$$

The set of utility matrices attainable under all feasible stochastic allocations (partitions of unity) is denoted by $\mathcal{V}_{\boldsymbol{\mu}}$. It is the set of all matrices $V_{\boldsymbol{\mu}}(f)$, where $f \in Z$:

$$\mathcal{V}_{\boldsymbol{\mu}} = \{V_{\boldsymbol{\mu}}(f) : f \in Z\}.$$

The set of utility matrices attainable under all feasible stochastic allocations that are simple functions is denoted by $\mathcal{V}^0_{\boldsymbol{\mu}}$. It is the set of all matrices $V_{\boldsymbol{\mu}}(f^0)$, where $f^0 \in Z^0$:

$$\mathcal{V}^0_{\boldsymbol{\mu}} = \{V_{\boldsymbol{\mu}}(f^0) : f^0 \in Z^0\}.$$

In other words, each component f_j^0 is a simple (step) function, and the components together form a partition of unity.

Finally, the set of utility matrices attainable under all feasible deterministic allocations is denoted by $\mathcal{V}_{\boldsymbol{\mu}}^*$. It is the set of all matrices $V_{\boldsymbol{\mu}}(f^*)$, where $f^* \in Z$ is deterministic:

$$\mathcal{V}_{\boldsymbol{\mu}}^* = \left\{ V_{\boldsymbol{\mu}}(f^F) : F \in X \right\},$$

where $f^F = (\mathbf{1}_{F_1}, \ldots, \mathbf{1}_{F_n})$ is the deterministic partition of unity associated with $F \in X$.

Theorem 5.1. *(Theorem 3 in DWW) If the preference measures $\mu_1, \ldots, \mu_n$ are finite, the simple range $\mathcal{V}_{\boldsymbol{\mu}}^0$ coincides with $\mathcal{V}_{\boldsymbol{\mu}}$: every utility matrix in $\mathcal{V}_{\boldsymbol{\mu}}$ can be represented as $V_{\boldsymbol{\mu}}(f^0)$ for some $f^0 \in Z^0$.*

Theorem 5.2. *(Theorem 4 in DWW) If the preference measures $\mu_1, \ldots, \mu_n$ are finite and atomless, the deterministic range $\mathcal{V}_{\boldsymbol{\mu}}^*$ coincides with the simple range $\mathcal{V}_{\boldsymbol{\mu}}^0$: every utility matrix in $\mathcal{V}_{\boldsymbol{\mu}}^0$ can be represented as $V_{\boldsymbol{\mu}}(f^*)$ for some deterministic partition of unity $f^* \in Z$.*

Observe that Theorems 5.1 and 5.2 imply that the sets $\mathcal{V}_{\boldsymbol{\mu}}$ and $\mathcal{V}_{\boldsymbol{\mu}}^*$ coincide if $\mu_1, \ldots, \mu_n$ are finite atomless measures. Recall that $\bar{f} \in Z$, hence $V_{\boldsymbol{\mu}}(\bar{f}) \in \mathcal{V}_{\boldsymbol{\mu}}$. Therefore, there exists a deterministic partition of unity $f^* = (\mathbf{1}_{F_1}, \ldots, \mathbf{1}_{F_n})$, where $F = (F_1, \ldots, F_n) \in X$, that gives agents the same utility matrix as $\bar{f}$:

$$V_{\boldsymbol{\mu}}(f^*) = V_{\boldsymbol{\mu}}(\bar{f}).$$

Now, let's consider the case when agents' measures $\{\mu_1, \ldots, \mu_N\}$ are not atomless. In that case, there is fairly little hope that 5.2 holds. Nevertheless, I show that the second best thing can be guaranteed. In the next theorem, in the spirit of the Birkhoff-von-Neumann theorem, I prove that every simple partition of unity is in the convex hull of the deterministic partitions of unity.

This result is stronger than utility-equivalence results, both universal and not. First and foremost, for every simple partition of unity, it gives a social planner a tool to implement an allocation by running a utility-equivalent lottery with finite support when the agents' utilities are linear. More so, the implementation result is invariant to other assumptions on agents' preferences, like going beyond linearity.

Theorem 5.3. *Let $\{f_j^0\}_{j \in J}$ be a family of simple functions that is a partition of unity. Then there exist finitely many families of indicator functions $\{f_{k,j}^*\}_{k \in K, j \in J}$ such that $f_{k,j}^* \geq 0, \sum_{j \in J} f_{k,j}^* = 1$ for every k and such that $(f_1^0, \ldots, f_N^0)$ is in the convex hull of $\{(f_{1,1}^*, \ldots f_{1,N}^*), \ldots, (f_{K,1}^*, \ldots f_{K,N}^*)\}$:*

$$\begin{bmatrix} f_1^0 \\ f_2^0 \\ \vdots \\ f_N^0 \end{bmatrix} = \sum_{k=1}^{K} a_k \begin{bmatrix} f_{1,k}^* \\ f_{2,k}^* \\ \vdots \\ f_{N,k}^* \end{bmatrix}$$

where $a_k \geq 0, \sum_{k \in K} a_k = 1$.

Proof. Here, I present an intuitive proof for the case of two agents. WLOG, the two simple functions can be defined on the same partition $\{E_i\}_{i\in n}$ of L. Let $f_1^0 = \sum_{i=1}^n \alpha_i \chi_{E_i}$ and $f_2^0 = \sum_{i=1}^n \beta_i \chi_{E_i}$ such that $\alpha_i + \beta_i = 1$. Order the coefficients for agent 1 in a non-decreasing order $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Then

$$f_1^0 = \alpha_1\chi_L + (\alpha_2 - \alpha_1)\chi_{L\setminus E_1} + (\alpha_3 - \alpha_2)\chi_{L\setminus(E_1\cup E_2)} + \cdots + (\alpha_n - \alpha_{n-1})\chi_{E_n} + (1 - \alpha_n)\chi_\emptyset$$

Now I claim that

$$f_2^0 = \alpha_1\chi_\emptyset + (\alpha_2 - \alpha_1)\chi_{E_1} + (\alpha_3 - \alpha_2)\chi_{E_1\cup E_2} + \cdots + (\alpha_n - \alpha_{n-1})\chi_{L\setminus E_n} + (1 - \alpha_n)\chi_L$$

Recall that for every $x \in L$, $f_2^0(x) = \beta_i$ for $E_i \ni x$. Using the form of f_2^0 above, observe for $x \in E_i$

$$f_2^0(x) = (\alpha_{i+1} - \alpha_i)\chi_{E_1\cup\cdots\cup E_i}(x) + (\alpha_{i+2} - \alpha_{i+1})\chi_{E_1\cup\cdots\cup E_{i+1}}(x) + \cdots + (1 - \alpha_n)\chi_L(x) =$$

$$= 1 - \alpha_i = \beta_i$$

Note that every pair of indicator functions is defined on a partition, and $\chi_{L\setminus(E_1\cup\cdots\cup E_m)} + \chi_{E_1\cup\cdots\cup E_m} = \chi_L$. Trivially, a telescoping sum $\sum_{i=1}^{n+1}(\alpha_i - \alpha_{i-1})$ with $\alpha_0 = 0$ and $\alpha_{n+1} = 1$ is equal to $\alpha_{n+1} = 1$.

The proof for an arbitrary finite number of agents is in the Appendix A. □

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A. Appendix

A.1 Measurability

Lemma A.1. *For every random partition A on a probability space $(\Omega, \mathcal{F}, P)$, the map $\Phi_A : \Omega \to X$ is measurable.*

Proof. Let $A = (A_1, \ldots, A_n)$ be a random partition on $(\Omega, \mathcal{F}, P)$. I first establish that $\Phi_A : \Omega \to X$ is measurable. By the definition of $\mathcal{M}_X$, it suffices to show that, for every finite positive measure μ on $(L, \mathcal{B})$ and every $j \in N$, the mapping

$$\omega \longmapsto \mu(F_j^\omega)$$

is $\mathcal{F}$-measurable.

Since $A_j \in \mathcal{B} \otimes \mathcal{F}$, its indicator function $\mathbf{1}_{A_j} : L \times \Omega \to \{0, 1\}$ is jointly measurable. Moreover,

$$\mu(F_j^\omega) = \int_L \mathbf{1}_{A_j}(x, \omega)\, d\mu(x).$$

By Tonelli's theorem, the right-hand side is a measurable function of ω. Therefore, Φ_A is measurable and induces a well-defined lottery $p_A = (\Phi_A)_\# P \in \Delta(X)$. □

A.2 Utility equivalence of lotteries and partitions of unity

Here is the full proof of theorem 4.3.

Proof. Fix $p \in \Delta(X)$, and let

$$m = \sum_{i \in N} \mu_i.$$

Since each μ_i is a finite positive measure, m is finite and $\mu_i \ll m$ for every $i \in N$.

For each $j \in N$, consider the point-membership function

$$g_j : L \times X \longrightarrow \{0, 1\}, \qquad g_j(x, F) = \mathbf{1}_{F_j}(x).$$

The function g_j is measurable in each argument separately, but it need not be jointly measurable. I therefore construct a jointly measurable approximation.

Because L is a standard Borel space, there exists an increasing sequence of finite measurable partitions

$$\mathcal{P}_k = \{E_{k,1}, \ldots, E_{k,r_k}\}$$

whose generated σ-algebras exhaust $\mathcal{B}$.

For every k and $j \in N$, define

$$g_{j,k}(x, F) = \sum_{\substack{1 \le r \le r_k \\ m(E_{k,r}) > 0}} \mathbf{1}_{E_{k,r}}(x) \frac{m(F_j \cap E_{k,r})}{m(E_{k,r})}.$$

Each $g_{j,k}$ is jointly measurable. Indeed, $x \mapsto \mathbf{1}_{E_{k,r}}(x)$ is $\mathcal{B}$-measurable, while

$$F \longmapsto m(F_j \cap E_{k,r})$$

is $\mathcal{M}_X$-measurable, since $E \mapsto m(E \cap E_{k,r})$ is a finite positive measure on $(L, \mathcal{B})$.

Moreover, for every fixed $F \in X$,

$$g_{j,k}(\cdot, F) = \mathbb{E}_m \left[\mathbf{1}_{F_j} \mid \sigma(\mathcal{P}_k) \right] \qquad m\text{-almost everywhere,}$$

where conditional expectation is understood with respect to the finite measure m.

By the martingale convergence theorem,

$$g_{j,k}(x, F) \longrightarrow \mathbf{1}_{F_j}(x) \qquad \text{for } m\text{-almost every } x \in L,$$

for every fixed $F \in X$.

Define

$$h_j(x, F) = \limsup_{k \to \infty} g_{j,k}(x, F).$$

Each h_j is jointly measurable, takes values in $[0, 1]$, and satisfies

$$h_j(\cdot, F) = \mathbf{1}_{F_j} \qquad m\text{-almost everywhere}$$

for every $F \in X$.

To ensure that the resulting functions form a partition of unity everywhere, let

$$D = \left\{ (x, F) \in L \times X : \sum_{j \in N} h_j(x, F) = 1 \right\}.$$

The set D is measurable. Define

$$\widetilde{g}_j(x, F) = \begin{cases} h_j(x, F), & (x, F) \in D, \\ 1, & (x, F) \notin D,\ j = 1, \\ 0, & (x, F) \notin D,\ j \neq 1. \end{cases}$$

The functions $\widetilde{g}_j$ are jointly measurable and satisfy

$$0 \leq \widetilde{g}_j(x, F) \leq 1, \qquad \sum_{j \in N} \widetilde{g}_j(x, F) = 1$$

for every $(x, F) \in L \times X$. Furthermore, for every $F \in X$,

$$\widetilde{g}_j(\cdot, F) = \mathbf{1}_{F_j} \qquad m\text{-almost everywhere.}$$

Now define

$$f_j(x) = \int_X \widetilde{g}_j(x, F)\, dp(F), \qquad j \in N.$$

By Tonelli's theorem, each f_j is $\mathcal{B}$-measurable. Also, for every $x \in L$,

$$0 \leq f_j(x) \leq 1$$

and

$$\sum_{j \in N} f_j(x) = \int_X \sum_{j \in N} \widetilde{g}_j(x, F)\, dp(F) = 1.$$

Thus, $f = (f_1, \ldots, f_n) \in Z$.

Finally, since $\mu_i \ll m$, the equality

$$\widetilde{g}_j(\cdot, F) = \mathbf{1}_{F_j}$$

holds μ_i-almost everywhere for every $F \in X$. Applying Tonelli's theorem gives

$$\begin{aligned}
\int_L f_j(x)\, d\mu_i(x) &= \int_L \int_X \widetilde{g}_j(x, F)\, dp(F)\, d\mu_i(x) \\
&= \int_X \int_L \widetilde{g}_j(x, F)\, d\mu_i(x)\, dp(F) \\
&= \int_X \int_L \mathbf{1}_{F_j}(x)\, d\mu_i(x)\, dp(F) \\
&= \int_X \mu_i(F_j)\, dp(F).
\end{aligned}$$

This holds for every $i, j \in N$. Therefore,

$$V_{\boldsymbol{\mu}}(f) = V_{\boldsymbol{\mu}}(p),$$

so f and p are utility-equivalent. □

□

A.3 Birkoff-von-Neumann

Theorem A.2. *Let $\{f_j^0\}_{j \in J}$ be a family of simple functions such that $f_j \geq 0, \sum_{j \in J} f_j = 1$. Then there exist finitely many families of indicator functions $\{f_{k,j}^*\}_{k \in K, j \in J}$ such that $f_{k,j}^* \geq 0, \sum_{j \in J} f_{k,j}^* = 1$ for every k and such that $(f_1^0, \ldots, f_N^0)$ is in the convex hull of $\{(f_{1,1}^*, \ldots f_{1,N}^*), \ldots, (f_{K,1}^*, \ldots f_{K,N}^*)\}$:*

$$\begin{bmatrix} f_1^0 \\ f_2^0 \\ \vdots \\ f_N^0 \end{bmatrix} = \sum_{k=1}^{K} a_k \begin{bmatrix} f_{1,k}^* \\ f_{2,k}^* \\ \vdots \\ f_{N,k}^* \end{bmatrix}$$

where $a_k \geq 0, \sum_{k \in K} a_k = 1$.

Proof. For any vector of simple functions $f = (f_1, \ldots f_N)$, there is a common partition $E_1, \ldots E_I$ of L such that every f_j is constant on every E_i. I denote values of the functions on each

element of the partition by f_j^i.

The proof goes as follows. In the next paragraph, I construct families of indicator functions corresponding to different partitions of L. Then, using Farkas' lemma, I claim there exists a vector of coefficients such that the vector f of simple functions is a convex combination of the chosen vectors of indicator functions.

Recall, N is the number of agents, and I is the number of elements in the partition. For every $m \in \{1, \ldots, \min(I, N)\}$, construct a coarser partition with m elements. How many partitions for a fixed m? Stirling number of the second kind $S(I, m)$. For every partition, choose m agents from N to give this partition, counting each permutation as a different subset of agents. How many permutations? $P(N, m) = \frac{N!}{(N-m)!}$. The claim now is that there exists $\{a_k^m\}$ such that

$$\begin{bmatrix} f_1^0 \\ f_2^0 \\ \vdots \\ f_N^0 \end{bmatrix} = \sum_{m=1}^{\min(I,N)} \sum_{k=1}^{S(I,m)\cdot P(N,m)} a_k^m \begin{bmatrix} \chi_{A_1^{k(m)}} \\ \vdots \\ \chi_{A_j^{k(m)}} \\ \vdots \\ \chi_{A_N^{k(m)}} \end{bmatrix}$$

Let me highlight important characteristics of the sets $(A_1^{k(m)}, \ldots, A_N^{k(m)})$. For every m, k, j, $A_j^{k(m)} = \bigcup E_i$ for some subset of $(1, \ldots, I)$, possibly empty. For every m, for every partition in $S(I, m)$ at most m out of N $A_j^{k(m)}$ are non-empty.

I establish the existence of non-negative coefficients $\{a_k^m\}$ that sum to one using Farkas' lemma. Note that in essence, one needs to prove that a system of linear equations $Pa = f$ admits a non-negative solution, where $a = \{a_k^m\}_{m \in M, k \in K}$, $f = \{f_j^i\}_{j \in J, i \in I}$ and P is a matrix of zeros and ones characterized below. I will also show below that $\sum_{m \in M} \sum_{k \in K} a_k^m = 1$ follows from the partition of unity constraint on f: $\sum_{j \in J} f_j^i = 1 \ \forall i \in I$.

The rows of P correspond to sets E_i and represent how many times a set E_i gets assigned to a particular agent. The index of a row is (j, i), and there are $J \times I$ rows in total. The entry (j, i) has 1 if in a column partition agent j is allocated a set E_i, otherwise it is zero.

The columns represent the assignment of the sets of the original partition $\{E_1, \ldots, E_N\}$ in different partitions. The columns are indexed by $(m, Par, Perm)$, where m represents how coarse a partition is; for every m, Par gives a particular partition with m elements and $Perm$ gives an assignment of m selected agents to the elements of the partition. $(Par, Perm)$ is indexed by $k(m)$. There are $\sum_m S(I, m) \times P(N, m)$ columns.

Let E_i be one of the subsets of the original partition. How many times does the agent j receive this subset in different partitions? Remove the set E_i from the pool of original subsets to be assigned to agents and repeat the procedure for the rest of $E_{k, k \neq i}$. Every row of P features $\sum_m^{min(I-1,N)} S(I-1, m) \cdot P(N, m)$ of 1s.

Every column contains I number of 1s, since every column represents how all $(E_1, \ldots, E_I)$ are allocated in a corresponding partition, corresponding permutation. Fix column $k(m)$, representing partition and permutation $k(m)$. For every block of I rows that corresponds to

agent j, for every row i, 1 represents that the set E_i is in a subset that is assigned to agent j. Hence, if the set E_i in partition $k(m)$ is allocated to agent j, there is 1 in the ith row of the agent j' block of rows, and there are zeros in the ith row of every other agent's block of rows. This description of P contains all the characteristics that are sufficient for the proof of the next claim.

Using Farkas' lemma, if there is no y such that $P^T y \geq 0$ and $f^T y < 0$, then I establish the existence of the coefficients in question.

Lemma A.3. *Let y be such that $P^T y \geq 0$. Then $f^T y \geq 0$.*

Proof. Observe that $f^T y = \sum_{i \in I} \left(\sum_{j \in J} f_j^i y_{ji} \right)$ is the sum of I convex combinations of $\{y_{ji}\}_{j \in J}$. If $f^T y < 0$, then there exists $\tilde{i}$ such that $\sum_{j \in J} f_j^{\tilde{i}} y_{j\tilde{i}} < 0$. WLOG, let's assume $\tilde{i} = 1$. Then there exists a subset of J agents such that $y_{\bar{j}}^1 < 0$ for every such $\bar{j}$. Also WLOG, let's assume agent 1 is in the subset J, so $y_{11} < 0$.

The idea of the proof is to use the fact that $P^T y \geq 0$. For every m, coarser partition and a particular permutation of agents $(m, k(m))$ $\sum_{j,i} P^T_{(m,k(m)),(j,i)} y_{ji} \geq 0$. Take a subset of these inequalities: select an inequality if for $(m, k(m))$, the column $(1,1)$ has 1 in it. I use this subset of inequalities to bound $|y_{11}|$ from above to ultimately show that $\sum_{i \geq 2} \left(\sum_{j \in J} f_j^i y_{ji} \right) \geq |y_{11}|$.

By varying the column $(j,1)$, we can apply the same construction and upper bound to any $y_{\bar{j}1}$ (and $y_{\bar{j}\tilde{i}}$). To see this, fix $\tilde{i}$. For every allocation $(m, k(m))$ in which agent k receives $E_{\tilde{i}}$, there is an allocation that is the same except $E_{\tilde{i}}$ goes to the agent l. Hence, the same $y_{j'i'}$ are picked for the agent l, except $y_{l\tilde{i}}$ is replaced with $y_{k\tilde{i}}$. Choosing $\tilde{i} = 1$ and picking agent $j = 1$ is, in fact, without loss.

Since $P^T y \geq 0$, list all the rows of P^T where the cell $(1,1)$ is equal to 1. The number of selected inequalities is equal to the number of times agent 1 gets E_1, $\sum_m^{min(I-1,N)} S(I,m) \cdot P(N,m)$. Every inequality will have $I-1$ of y_{ji} on the left-hand side. Whenever agent 1 gets E_1, no other agent gets it, so inequalities will not feature $y_{j1}, j \neq 1$.

The goal is to show that

$$\sum_{j \in J} f_j^2 y_j^2 \geq |y_1^1| - \sum_{i \geq 3} \sum_{j \in J} f_j^i y_j^i$$

The procedure is iterative. The main idea is that there are enough inequalities to show first that $\sum_{j \in J} f_j^2 y_j^2 \geq |y_1^1| - c(I)$, where $c(I)$ is some constant. Iteratively proceed to "extract" elements indexed by I from $c(I)$, getting that $\sum_{j \in J} f_j^2 y_j^2 \geq |y_1^1| - c(I-1) - \sum_{j \in J} f_j^I y_j^I$ and so on.

Observe that for every $j \in J$ for the constant $c(I)$ that is independent of j, there exists a partition and a permutation to support

$$y_j^2 \geq |y_1^1| - c(I)$$

There are J inequalities above, and they all hold for any constant $c(I)$ of the type $c(I) = \sum_{i \neq 1,2} y^i_{j(i)}$, $j(i) \in J$ for every i. How many constants $c(I)$ of this type are there? We need to allocate $I-2$ sets to N agents, giving $\sum_{m=1}^{min(I-2,N)} S(I-2,m) \cdot P(N,m)$ constants.

In particular, for every $j \in J$

$$\sum_{j \in J} f_j^2 y_j^2 \geq |y_1^1| - c(I-1) - y_j^I$$

where the inequality holds for any constant of type $c(I-1) = \sum_{i \neq 1,2,I} y^i_{j(i)}$, $j(i) \in J$ for every i. Similarly to the previous paragraph, there are $\sum_{m=1}^{min(I-3,N)} S(I-3,m) \cdot P(N,m)$ constants.

Therefore

$$\sum_{j \in J} f_j^2 y_j^2 \geq |y_1^1| - c(I-1) - \sum_{j \in J} f_j^I y_j^I$$

Iterating, for every $j \in J$

$$\sum_{j \in J} f_j^2 y_j^2 \geq |y_1^1| - c(I-2) - y_j^{I-1} - \sum_{j \in J} f_j^I y_j^I$$

for $c(I-2) = \sum_{i \neq 1,2,I,I-1} y^i_{j(i)}$, $j(i) \in J$ for every i.

Therefore

$$\sum_{j \in J} f_j^2 y_j^2 \geq |y_1^1| - c(I-2) - \left(\sum_{j \in J} f_j^{I-1} y_j^{I-1} + \sum_{j \in J} f_j^I y_j^I \right)$$

Proceeding iteratively, I establish the desired result given that there are enough inequalities to support the claims above.

Lemma A.4. *Let ξ be a natural number such that $\xi < I$. For every such ξ*

$$\sum_{m=1}^{min(I-\xi,N)} S(I-\xi,m) \cdot P(N,m) = N \times \sum_{m=1}^{min(I-(\xi+1),N)} S(I-(\xi+1),m) \cdot P(N,m)$$

Proof. Using the recurrence relation of the Stirling numbers of the second kind,

$$S(I-\xi,m) = mS(I-(\xi+1),m) + S(I-(\xi+1),m-1)$$

I get

$$\sum_{m=1}^{min(I-\xi,N)} S(I-\xi,m) \cdot P(N,m) = \sum_{m=1}^{min(I-\xi,N)} mS(I-\xi,m) \cdot P(N,m) +$$

$$+ \sum_{m=1}^{min(I-\xi,N)} S(I-(\xi+1),m-1) \cdot P(N,m)$$

Note that $S(n,m) = 0$ where $m > n$, so for all $m > I-(\xi+1)$, $S(I-(\xi+1),m) = 0$. Since N is a natural number, $\sum_{m=1}^{min(I-\xi,N)} mS(I-(\xi+1),m) \cdot P(N,m) = \sum_{m=1}^{min(I-(\xi+1),N)} mS(I-(\xi+1),m) \cdot P(N,m)$. For the second term in the sum on the right-hand side, re-labeling produces $\sum_{m=0}^{min(I-\xi,N)-1} S(I-(\xi+1),m) \cdot P(N,m+1)$. $S(n,0) = 0$ for any $n > 0$, and

$P(N, m+1) = (N-m)P(N,m)$. Summing the two terms again produces

$$\sum_{m=1}^{min(I-(\xi+1),N)} mS(I-(\xi+1),m)\cdot P(N,m) + \sum_{m=1}^{min(I-\xi,N)-1} S(I-(\xi+1),m)\cdot P(N,m)(N-m) =$$

$$= N \times \sum_{m=1}^{min(I-(\xi+1),N)} S(I-(\xi+1),m)\cdot P(N,m)$$

since if $N \geq 1-\xi$, $min(1-\xi,N)-1 = I-(\xi+1)$, and if $N < 1-\xi$, $min(I-(\xi+1),N) = min(I-\xi,N) = N$. □

The claim above establishes the fact that there are just enough inequalities. Every step in the iterative process uses all inequalities from the previous step, and supports the idea used in the construction of the matrix P, that is, allocating subsets of the original partition to agents in all possible ways is necessary to apply Farkas' lemma. □

The last claim to prove is $\sum_{m\in M}\sum_{k(m)\in K(m)} a^m_{k(m)} = 1$. To this end, I show that for any i, $\sum_{m\in M}\sum_{k(m)\in K(m)} a^m_{k(m)} = \sum_{j\in J} f^i_j$. Fix i. By construction $\sum_{k,m(k)} P_{(ji),(m,k(m))} a^m_{k(m)} = f^i_j$. Observe that the 1s in a row (j,i) reflect the partitions in which agent j gets E_i. For every column, every original subset E_i, if agent k gets E_i, then no other agent is allocated E_i. Thus, every $a^m_{k(m)}$ is present at most once in the sum $\sum_{j\in J}\sum_{k,m(k)} P_{(ji),(m,k(m))} a^m_{k(m)}$. On the other hand, since every $m, k(m)$ assigns an agent to all original subsets, including E_i, all $a^m_{k(m)}$ are featured in $\sum_{j\in J}\sum_{k,m(k)} P_{(ji),(m,k(m))} a^m_{k(m)}$. □

B. Counter-example for the universal utility equivalence

This counter-example constructs a lottery over deterministic allocations that does not admit a universally utility-equivalent partition of unity. Let

$$L = [0,1], \qquad \mathcal{B} = \mathcal{B}([0,1]),$$

and consider two agents, $N = \{1,2\}$.

Choose a non-Borel subset $S \subseteq L$ of cardinality $\aleph_1$, and enumerate its elements as

$$S = \{x_\alpha : \alpha < \omega_1\},$$

where ω_1 is the first uncountable ordinal. Such a set exists in ZFC.

For every $\alpha < \omega_1$, define

$$E_\alpha = \{x_\beta : \beta < \alpha\}.$$

Since every $\alpha < \omega_1$ is countable, E_α is a countable subset of L and therefore belongs to $\mathcal{B}$.

Consequently,

$$F^\alpha = (E_\alpha, L \setminus E_\alpha) \in X$$

is a feasible deterministic allocation.

Now take $\Omega = \omega_1$, equipped with the countable–cocountable σ-algebra

$$\mathcal{F} = \{E \subseteq \omega_1 : E \text{ is countable or } \omega_1 \setminus E \text{ is countable}\}.$$

Define a probability measure P on $(\Omega, \mathcal{F})$ by

$$P(E) = \begin{cases} 0, & E \text{ is countable,} \\ 1, & \omega_1 \setminus E \text{ is countable.} \end{cases}$$

Consider the mapping

$$\Phi : \Omega \longrightarrow X, \qquad \Phi(\alpha) = F^\alpha.$$

Recall that $\mathcal{M}_X$ is generated by the mappings

$$F \longmapsto \mu(F_j), \qquad \mu \in \mathcal{M}_f(L), \quad j \in N.$$

Fix an arbitrary finite positive measure $\mu \in \mathcal{M}_f(L)$. Since μ is finite, the set

$$D_\mu = \{x \in L : \mu(\{x\}) > 0\}$$

is at most countable.

Therefore, there exists a countable ordinal $\alpha_\mu < \omega_1$ such that

$$D_\mu \cap S \subseteq E_{\alpha_\mu}.$$

For every $\alpha \geq \alpha_\mu$, the difference $E_\alpha \setminus E_{\alpha_\mu}$ is countable and contains no point of positive μ-mass. Hence,

$$\mu(E_\alpha) = \mu(E_{\alpha_\mu}) \qquad \text{for all } \alpha \geq \alpha_\mu.$$

Thus, the mapping

$$\alpha \longmapsto \mu(E_\alpha)$$

is eventually constant. In particular, it is $\mathcal{F}$-measurable, since every initial segment of ω_1 is countable.

The same conclusion holds for

$$\alpha \longmapsto \mu(L \setminus E_\alpha) = \mu(L) - \mu(E_\alpha).$$

It follows that Φ is measurable. Consequently, its pushforward

$$p = \Phi_\# P$$

is a well-defined lottery in $\Delta(X)$.

For each $x \in L$, consider the probability that agent 1 receives x under p:

$$q(x) = p\Big(\{F \in X : x \in F_1\}\Big).$$

If $x \notin S$, then $x \notin E_\alpha$ for every $\alpha < \omega_1$. Therefore,

$$q(x) = 0.$$

If $x = x_\beta \in S$, then

$$x_\beta \in E_\alpha \quad \Longleftrightarrow \quad \alpha > \beta.$$

Since the set $\{\alpha < \omega_1 : \alpha > \beta\}$ is cocountable in ω_1, it has P-measure one. Hence $q(x_\beta) = 1$. Combining the two cases gives

$$q(x) = \mathbf{1}_S(x) \qquad \text{for every } x \in L.$$

Now I am in postion to prove the claim. Suppose, toward a contradiction, that there exists a partition of unity $f \in Z$ that is universally utility-equivalent to p.

By definition, for every finite positive measure $\mu \in \mathcal{M}_f(L)$,

$$\int_X \mu(F_1)\, dp(F) = \int_L f_1\, d\mu.$$

Fix an arbitrary $x \in L$ and take $\mu = \delta_x$, the Dirac measure at x. Then

$$\begin{aligned} f_1(x) &= \int_L f_1\, d\delta_x \\ &= \int_X \delta_x(F_1)\, dp(F) \\ &= p\Big(\{F \in X : x \in F_1\}\Big) \\ &= q(x) \\ &= \mathbf{1}_S(x). \end{aligned}$$

Since x was arbitrary,

$$f_1 = \mathbf{1}_S.$$

This contradicts the requirement that f_1 be $\mathcal{B}$-measurable.

Therefore, no partition of unity is universally utility-equivalent to p.